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On the Gelfand–Graev Representations of a Reductive Group over a Finite Field

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1. INTRODUCTION

Let G be a connected, reductive algebraic group, defined over a finite field $\mathbb{F}_q$, with Frobenius map F. We shall be concerned with the Gelfand–Graev representations Γ of the finite group G^F. The irreducible components of Γ were described, for groups with connected centers, by Deligne and Lusztig [5, Chap. 10], in terms of the virtual representations $\{R^G_{T,\theta}\}$. In the general case, the Gelfand–Graev representations were parametrized, and decomposed into irreducible components, by Digne and Michel [6, Sect. 14] and by Digne *et al.* [7].

In this paper, we study the irreducible representations of the Hecke algebra $\mathscr{H}$ (or the G^F-endomorphism algebra) of a Gelfand–Graev representation Γ of G^F. Each Gelfand–Graev representation is multiplicity free, and its Hecke algebra $\mathscr{H}$ is commutative, by [13, Theorem 49].

The first result (Theorem 3.1) describes the irreducible representations of $\mathscr{H}$. These are parametrized by pairs (T, θ), each consisting of an F-stable maximal torus T and an irreducible character θ of T^F. Each representation $f_{T,\theta}$ of $\mathscr{H}$ corresponds to the unique irreducible character $\chi_{T,\theta}$ which occurs with nonzero multiplicity in both Γ and $R^G_{T,\theta}$.

The main result is a factorization theorem (Theorem 4.2), which states that each representation $f_{T,\theta}$ has a factorization $f_{T,\theta} = \hat{\theta} \circ f_T$, with a homomorphism of algebras f_T from $\mathscr{H}$ to the group algebra $\bar{\mathbb{Q}}_l T^F$ of the maximal torus T^F, followed by the extension $\hat{\theta}$ of the character θ of T^F to an irreducible representation of the group algebra. The homomorphism f_T is independent of θ, and is given by an explicit formula involving Green

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functions $\{Q_T^H\}$ of connected reductive F-stable subgroups H of G, and an irreducible linear character ψ of U^F such that $\Gamma = \psi^{G^F}$.

In Section 5, the homomorphisms f_T are computed, in the case $G = SL_2$. The resulting irreducible representations $f_{T,\theta}$ of $\mathscr{H}$ are given by formulas first obtained by Gelfand and Graev [9], who called them Bessel functions over finite fields. We also give a direct proof in Section 5, independent of the Deligne–Lusztig theory, that the formula for the map $f_T: \mathscr{H} \to \bar{\mathbb{Q}}_l T^F$ gives a homomorphism of algebras, for the twisted torus T of SL_2. The proof is based on identities for exponential sums over finite fields. The corresponding representations $f_{T,\theta}$ of $\mathscr{H}$, in this case, correspond to cuspidal characters of SL_2.

The results in Sections 2–4 were first proved for groups with connected centers. I am indebted to F. Digne and J. Michel for explaining to me how to modify the proofs to handle the general case, using the analysis in [6, 7] of the Gelfand–Graev representations of arbitrary reductive groups.

Notation

$\mathbb{F}_q$ denotes the finite field with q elements, of characteristic p.

The set-up G, $F: G \to G$, and G^F introduced earlier in this section, are explained in more detail in [1, 5, 6, 11].

For a prime $l \neq p$, $\bar{\mathbb{Q}}_l$ is an algebraic closure of the l-adic field.

Representations and characters of finite groups will be taken in the field $\bar{\mathbb{Q}}_l$.

For a finite group H, and class functions f and g on H, with values in a cyclotomic field contained in $\bar{\mathbb{Q}}_l$, $\langle f, g \rangle$ or $\langle f, g \rangle_H$ denotes the inner product $|H|^{-1} \sum_{x \in H} f(x)\,\overline{g(x)}$, where $a \to \bar{a}$ is the automorphism of $\bar{\mathbb{Q}}_l$ extending the map $\xi \to \xi^{-1}$, for roots of unity ξ.

The $\mathbb{F}_q$-rank of G is denoted by $\sigma(G)$, and ε_G denotes $(-1)^{\sigma(G)}$. Similarly $\varepsilon_T = (-1)^{\sigma(T)}$, for an F-stable maximal torus T.

Irr H denotes the set of irreducible characters afforded by simple $\bar{\mathbb{Q}}_l H$-modules, for a finite group H.

$Z(G)$ denotes the center of G, and $Z(G)^\circ$ its connected component of the identity.

$C_G(x)$ denotes the centralizer of x in G, and $C_G(x)^\circ$ its connected identity component.

2. Preliminary Results

2A. We first describe, following [6] (or [7]), the parametrization of the Gelfand–Graev characters of G^F. We choose a pair (T_0, B_0) consisting of an F-stable maximal torus T_0 and an F-stable Borel subgroup B_0 containing T_0. Then $W = N_G(T_0)/T_0$ is the Weyl group of G. Let Φ be the root system of G with respect to T_0, Φ^+ the set of positive roots corresponding to B_0, and Π the set of simple roots in Φ^+. Let U_0 be the unipotent radical of B_0, and let U_0^* be the subgroup of U_0 generated by the root subgroups corresponding to the nonsimple roots. Then U_0/U_0^* is abelian, and is isomorphic to the direct product of the simple root subgroups $\{U_\alpha\}_{\alpha\in\Pi}$. The Frobenius endomorphism F permutes the simple roots. Let I be the set of F-orbits on Π, and for each $i \in I$, let $U_i = \prod_{\alpha\in i} U_\alpha$. Then $U_0/U_0^* \cong \prod_{i\in I} U_i$. Since the subgroups involved are all F-stable, we have

$$U_0^F/U_0^{*F} \cong \prod_{i\in I} U_i^F.$$

Each linear character ψ of U_0^F which is trivial on U_0^{*F} is called *nondegenerate* if $\psi \mid U_i^F \neq 1$ for each orbit $i \in I$.

The *Gelfand–Graev characters* of G^F are the induced characters ψ^{G^F} $(= \operatorname{ind}_{U_0^F}^{G^F} \psi)$ from the nondegenerate linear characters of U_0^F.

The subgroup T_0^F of G^F normalizes U_0^F and the subgroups U_i^F. It acts by conjugation on the set of nondegenerate linear characters of U_0^F. By [6, Prop. 14.28], the T_0^F-orbits of nondegenerate linear characters of U_0^F are in one to one correspondence with the elements of $H^1(F, Z(G))$, which is the set of F-conjugacy classes of the center $Z(G)$ (see also [7, Sect. 2] and [11], which is a source of some of these ideas).

In [6, Def. 14.29 and the preceding discussion], a nondegenerate linear character ψ_z of U_0^F is defined for each element $z \in H^1(F, Z(G))$. Each Gelfand–Graev character Γ coincides with $\Gamma_z = \psi_z^{G^F}$ for a unique choice of $z \in H^1(F, Z(G))$. Moreover, the Gelfand–Graev characters Γ_z are multiplicity free, by [13, Theorem 49]. (In case $Z(G)$ is connected, $H^1(F, Z(G)) = 1$, and there is just one T_0^F-orbit of nondegenerate characters of U_0^F. Therefore there is a unique Gelfand–Graev character, in this case. See [1, Sect. 8.1] for further details.)

Let (T, θ) be a pair consisting of an F-stable maximal torus and an irreducible character θ of T^F. We let $R_{T,\theta}^G$ denote the virtual character of G^F associated with the pair (T, θ) (as in Deligne and Lusztig [5]; see also [1, 6, 12]). The main result of this subsection is:

(2.1) THEOREM. *Let Γ_z be a fixed Gelfand–Graev character of G^F, for some $z \in H^1(F, Z(G))$, and let (T, θ) be a pair, as above. Then there exists a unique irreducible character $\chi_{T,\theta,z}$ of G^F such that*

$$\langle \chi_{T,\theta,z}, \Gamma_z \rangle \neq 0 \qquad \textit{and} \qquad \langle \chi_{T,\theta,z}, R_{T,\theta}^G \rangle \neq 0.$$

Each irreducible character occurring with positive multiplicity in Γ_z coincides with $\chi_{T,\theta,z}$ for some pair (T, θ).

This result follows directly from various theorems proved in [6]. We sketch the ideas involved in the proof. In case $Z(G)$ is connected, the result follows from [5, Theorem 10.7] (see also [1, p. 379]). For the general case, let G^* be a connected, reductive group with Frobenius endomorphism F^*, such that the pairs (G, F) and (G^*, F^*) are in duality ([6, Def. 13.10]). By [6, Prop. 13.13], the G^F-conjugacy classes of pairs (T, θ), as above, are in bijection with the G^{*F^*}-conjugacy classes of pairs (T^*, s), with s a semisimple element of G^{*F^*}, and T^* an F^*-stable maximal torus of G^* containing s. Because of this fact, we can use the notation $R^G_{T^*,s}$ for $R^G_{T,\theta}$.

For Γ_z as in the statement of the theorem, we have

$$\Gamma_z = \sum_{(s)} \chi_{s,z}$$

for certain irreducible characters $\chi_{s,z}$ of G^F parametrized by the semisimple conjugacy classes (s) of G^{*F^*}, by [6, Theorem 14.49]. Each character $\chi_{s,z}$ is characterized as the unique irreducible character of G^F occurring with positive multiplicity in both Γ_z and a certain character $\chi_{(s)}$ of G^F, defined by

$$\chi_{(s)} = |W^\circ(s)|^{-1} \sum_{w \in W^\circ(s)} \varepsilon_G \varepsilon_{T^*_w} R^G_{T^*_w, s},$$

where $W^\circ(s)$ is the Weyl group of the reductive group $C_{G^*}(s)^\circ$, and T^*_w is a maximal torus in G^* of type w, for $w \in W^\circ(s)$. (See [6, Prop. 14.48] for a proof that the class function $\chi_{(s)}$ is a character of G^F.)

For a given pair (T, θ), let $R^G_{T,\theta} = R^G_{T^*,s}$, for a semisimple element $s \in G^{*F^*}$ and an F^*-stable maximal torus T^* of G^* containing s. Since $\chi_{s,z}$ is a component of $\chi_{(s)}$, we have $\langle \chi_{s,z}, R^G_{T^*,s} \rangle \neq 0$ by the proof of [6, Prop. 14.51]. On the other hand, no character $\chi_{s',z}$ occurs in $R^G_{T^*,s}$ if the G^{*F^*}-classes (s) and (s') are distinct, otherwise $R^G_{T^*,s}$ and $R^G_{T^*,s'}$ have a common irreducible component, contrary to [6, Prop. 14.51]. It follows that $\chi_{s,z}$ is the unique irreducible character occurring in both Γ_z and $R^G_{T^*,s}$. Since $R^G_{T,\theta} = R^G_{T^*,s}$, the character $\chi_{T,\theta,z} = \chi_{s,z}$ satisfies the requirements of the theorem. The last statement follows from the formula $\Gamma_z = \sum \chi_{s,z}$.

2B. In this subsection, we recall the notation and some of the basic facts concerning Hecke algebras of induced linear characters, from [4, Sect. 11D].

Let G be a finite group, U a subgroup, and ψ a linear character of U, with values in some algebraically closed field K of characteristic zero. Let

$$\mathbf{e}_\psi = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u;$$

then $\mathbf{e}_\psi$ is the primitive central idempotent in the group algebra KU affording the representation ψ. The *Hecke algebra* $\mathscr{H}_\psi$ of the induced character ψ^G is defined by

$$\mathscr{H}_\psi = \mathbf{e}_\psi KG \mathbf{e}_\psi;$$

it is a subalgebra of the group algebra KG with identity element $\mathbf{e}_\psi$, and is antiisomorphic to the KG-endomorphism algebra of the induced KG-module affording ψ^G.

Let N be a cross section of the double cosets $U\backslash G/U$. Let $n \in N$, and let

$$\operatorname{ind} n = |UnU|/|U| = |U : U \cap {}^nU|.$$

If $\psi = {}^n\psi$ on $U \cap {}^nU$, then $\mathbf{e}_\psi n \mathbf{e}_\psi \neq 0$, and the corresponding elements

$$\mathbf{c}_n = \operatorname{ind} n \mathbf{e}_\psi n \mathbf{e}_\psi$$

of $\mathscr{H}_\psi$ are the *standard basis elements* of $\mathscr{H}_\psi$.

The connections between irreducible components of the induced character ψ^G and irreducible characters of the Hecke algebra $\mathscr{H}_\psi$ are summarized as follows.

(2.2) Proposition. (i) *There is a bijection, given by restriction from KG to $\mathscr{H}_\psi$, from the set of irreducible characters χ of G such that $\langle \chi, \psi^G \rangle \neq 0$ to the set of all irreducible characters of $\mathscr{H}_\psi$.*

(ii) *The primitive central idempotents of $\mathscr{H}_\psi$ are the elements $\{\mathbf{e}_\psi \varepsilon_\chi\}$, where ε_χ is a primitive central idempotent of KG associated with an irreducible character χ such that $\langle \chi, \psi^G \rangle \neq 0$.*

(iii) *The induced character ψ^G is multiplicity free if and only if the Hecke algebra $\mathscr{H}_\psi$ is commutative.*

(iv) *Let $\langle \chi, \psi^G \rangle \neq 0$, and let $\varphi = \chi|_{\mathscr{H}_\psi}$, as in* (i). *Then*

$$\deg \chi = \chi(1) = \langle \chi, \psi^G \rangle\, |G : U| \left(\sum_n (\operatorname{ind} n)^{-1} \varphi(\mathbf{c}_n)\, \varphi(\hat{\mathbf{c}}_n) \right)^{-1},$$

where the sum is taken over the standard basis elements $\{\mathbf{c}_n\}$, and $\hat{\mathbf{c}}_n =$ $\operatorname{ind} n \mathbf{e}_\psi n^{-1} \mathbf{e}_\psi$.

(v) *Let χ, ψ be as in* (iv). *Let $g \in G$. Then*

$$\chi(g)\, \chi(1)^{-1} = |C_G(g)|\, |G|^{-1} \sum_n \varphi(\mathbf{c}_n)\, \psi(\mathfrak{C}_n),$$

where

$$\mathfrak{C}_n = \sum_{y \in U \cap \mathfrak{C}n^{-1}} y,$$

and $\mathfrak{C}$ is the conjugacy class of G containing g.

Parts (i)–(iv) are proved in [4, Props. 11.32 and 11.26]. Part (v) is due to Ree, and is stated in [2, Prop. 7.1]. For a proof of (v), see [3, Prop. 2.5].

3. The Irreducible Representations of the Hecke Algebras of Gelfand–Graev Representations

Let $\Gamma_z = \psi_z^{G^F}$ be a fixed Gelfand–Graev character of G^F, corresponding to an element $z \in H^1(F, Z(G))$, as in Sect. 2A. Its Hecke algebra is

$$\mathscr{H}_z = \mathbf{e}_z \bar{\mathbb{Q}}_l G^F \mathbf{e}_z,$$

where

$$\mathbf{e}_z = |U_0^F|^{-1} \sum_{u \in U_0^F} \psi_z(u^{-1}) u.$$

Since Γ_z is multiplicity free (see Section 2A), $\mathscr{H}_z$ is a commutative split semisimple algebra over $\bar{\mathbb{Q}}_l$.

We first introduce some more notation. Let α be a function on G^F with values in $\bar{\mathbb{Q}}_l$. We define a corresponding element in the group algebra $\bar{\mathbb{Q}}_l G^F$ by

$$\mathbf{a} = \sum_{g \in G^F} \alpha(g^{-1}) g.$$

If $\mathbf{a}$ and $\mathbf{b}$ are elements of $\bar{\mathbb{Q}}_l G^F$ corresponding to the functions α and β, then their product $\mathbf{ab}$ in the group algebra corresponds to the convolution product of α and β. For an irreducible character χ of G^F, the corresponding element $\mathbf{x} = \sum_{g \in G^F} \chi(g^{-1}) g$ is a multiple by an element of $\mathbb{Q}$ of the primitive central idempotent in $\bar{\mathbb{Q}}_l G^F$ associated with χ, by [4, Prop. 9.21].

The next result describes the Wedderburn components, and the irreducible representations associated with them, of the Hecke algebra $\mathscr{H}_z$ of Γ_z.

(3.1) Theorem. *Let $\Gamma_z = \psi_z^{G^F}$, $\mathscr{H}_z = \mathbf{e}_z \bar{\mathbb{Q}}_l G^F \mathbf{e}_z$, and let (T, θ) be a pair consisting of an F-stable maximal torus T and an irreducible character θ of T^F. Let $\mathbf{r}_{T,\theta}$ and $\mathbf{x}_{T,\theta,z}$ be the elements of the group algebra $\bar{\mathbb{Q}}_l G^F$ corresponding to the virtual character $R_{T,\theta}^G$, and the irreducible character $\chi_{T,\theta,z}$*

(defined in Theorem 2.1), respectively. Let $\mathbf{a}_{T,\theta,z}$ be the element of the group algebra defined by

$$\mathbf{a}_{T,\theta,z} = \mathbf{e}_z \mathbf{r}_{T,\theta}.$$

Then the following statements hold.

(i) *$\mathbf{a}_{T,\theta,z}$ is a nonzero element of $\mathscr{H}_z$, and $\bar{\mathbb{Q}}_l \mathbf{a}_{T,\theta,z}$ is a Wedderburn component of the commutative algebra $\mathscr{H}_z$.*

(ii) *The irreducible representation $f_{T,\theta,z}$ of $\mathscr{H}_z$ associated with $\mathbf{a}_{T,\theta,z}$, and defined by*

$$\mathbf{c}\cdot\mathbf{a}_{T,\theta,z} = f_{T,\theta,z}(\mathbf{c})\,\mathbf{a}_{T,\theta,z}, \qquad \textit{for} \quad \mathbf{c}\in\mathscr{H}_z,$$

corresponds (in the sense of (2.2i)) with the irreducible character $\chi_{T,\theta,z}$.

(iii) *Let (T', θ') be another pair, and $f_{T',\theta',z}$ the irreducible representation of $\mathscr{H}_z$ associated with it. Then $f_{T,\theta,z} = f_{T',\theta',z}$ if and only if the pairs (T, θ) and (T', θ') correspond to the same semisimple conjugacy class (s) in G^{*F^*}.*

(iv) *The Wedderburn decomposition of $\mathscr{H}_z$ is given by*

$$\mathscr{H}_z = \bigoplus_{(T,\theta)} \bar{\mathbb{Q}}_l \mathbf{a}_{T,\theta,z},$$

where the sum is taken over pairs (T, θ) corresponding to the semisimple classes of G^{*F^*}. The irreducible representations $\{f_{T,\theta,z}\}$ of $\mathscr{H}_z$ associated with them form a complete set.

Proof. By Theorem 2.1, the expansion of $R^G_{T,\theta}$ in terms of irreducible characters of G^F involves $\chi_{T,\theta,z}$ with a nonzero coefficient, and no other irreducible character ξ of G^F such that $\langle \xi, \Gamma_z\rangle \neq 0$. It follows that

$$\mathbf{r}_{T,\theta} = k\mathbf{x}_{T,\theta,z} + \mathbf{y},$$

with $k \neq 0$ in $\bar{\mathbb{Q}}_l$, and $\mathbf{y}$ a linear combination of primitive central idempotents in $\bar{\mathbb{Q}}_l G^F$ corresponding to irreducible characters ζ such that $\langle \zeta, \Gamma_z\rangle = 0$. Then

$$\mathbf{a}_{T,\theta,z} = \mathbf{e}_z \mathbf{r}_{T,\theta} = k\mathbf{e}_z \mathbf{x}_{T,\theta,z} \neq 0 \tag{3.2}$$

since $\mathbf{e}_z\mathbf{x}_{T,\theta,z} \neq 0$ and $\mathbf{e}_z\mathbf{y} = 0$ by Proposition 2.2ii, and the preceding remarks. Since $\mathbf{x}_{T,\theta,z}$ is a nonzero multiple of the primitive central idempotent corresponding to $\chi_{T,\theta,z}$, it follows that $\mathbf{e}_z\mathbf{x}_{T,\theta,z}$, and hence $\mathbf{a}_{T,\theta,z}$, are multiples of a primitive central idempotent in $\mathscr{H}_z$. Since $\mathscr{H}_z$ is commutative,

$\mathscr{H}_z \mathbf{a}_{T,\theta,z} = \bar{\mathbb{Q}}_l \mathbf{a}_{T,\theta,z}$ is a Wedderburn component of $\mathscr{H}_z$. Moreover, $\mathbf{a}_{T,\theta,z}$ affords the irreducible representation $f_{T,\theta,z}$ of $\mathscr{H}_z$ of degree one, such that

$$\mathbf{c}\mathbf{a}_{T,\theta,z} = f_{T,\theta,z}(\mathbf{c})\,\mathbf{a}_{T,\theta,z} \qquad \text{for all} \quad \mathbf{c} \in \mathscr{H}_z.$$

By the previous discussion $\mathbf{a}_{T,\theta,z}$ is a multiple of the primitive idempotent $\mathbf{e}_z\varepsilon$ in $\mathscr{H}_z$, where ε is the primitive central idempotent in $\bar{\mathbb{Q}}_l G^F$ associated with $\chi_{T,\theta,z}$. Therefore the irreducible representation $f_{T,\theta,z}$ corresponds to $\chi_{T,\theta,z}$ in the sense of Proposition 2.2i. This completes the proof of parts (i) and (ii).

For the proof of part (iii), $f_{T,\theta,z} = f_{T',\theta',z}$ if and only if $\chi_{T,\theta,z} = \chi_{T',\theta',z}$. From the proof of Theorem 2.1, we have

$$\chi_{T,\theta,z} = \chi_{s,z} \qquad \text{and} \qquad \chi_{T',\theta',z} = \chi_{s',z},$$

for semisimple elements s and s' in G^{*F^*}, such that $R^G_{T,\theta} = R^G_{T^*,s}$ and $R^G_{T',\theta'} = R^G_{T'^*,s'}$. Then $\chi_{s,z} = \chi_{s',z}$ only if $R^G_{T^*,s}$ and $R^G_{T'^*,s'}$ have a common irreducible component. By [6, Theorem 14.51], this occurs only if s and s' are conjugate in G^{*F^*}. On the other hand, if s and s' are rationally conjugate, then $\chi_{s,z} = \chi_{s',z}$ by [6, Theorem 14.49]. This completes the proof of part (iii). Part (iv) follows from parts (i)–(iii) and Theorem 2.1.

Remarks. (i) Let

$$\mathbf{a}_{T,\theta,z} = \sum_{g \in G^F} \alpha_{T,\theta,z}(g^{-1})\,g$$

be the element of $\mathscr{H}_z$ introduced in Theorem 3.1. Then the function $\alpha_{T,\theta,z}$ is given by the formula

$$\alpha_{T,\theta,z}(g) = |U_0^F|^{-1} \sum_{u \in U_0^F} \psi_z(u^{-1})\,R^G_{T,\theta}(ug).$$

This formula, with $R^G_{T,\theta}$ replaced by an irreducible character χ, was first introduced, in case $G = GL_n$, by S. I. Gelfand [8, Sect. 4], and called the *Bessel function* associated with χ, if different from zero.

(ii) The primitive idempotents in the commutative algebra $\mathscr{H}_z$ are the elements $\{f_{T,\theta,z}(\mathbf{a}_{T,\theta,z})^{-1}\,\mathbf{a}_{T,\theta,z}\}$, by parts (i) and (ii) of Theorem 3.1.

4. A Factorization Theorem

By Theorem 3.1, each irreducible representation $f_{T,\theta,z}$ of $\mathscr{H}_z$ corresponds to the unique irreducible character $\chi_{T,\theta,z}$ of G^F which occurs as a common constituent of the virtual character $R^G_{T,\theta}$ and the Gelfand–Graev character

Γ_z. The main result of this section asserts that the homomorphism $f_{T,\theta,z}\colon \mathcal{H}_z \to \bar{\mathbb{Q}}_l$ can be factored through the group algebra $\bar{\mathbb{Q}}_l T^F$. The resulting homomorphism $f_{T,z}\colon \mathcal{H}_z \to \bar{\mathbb{Q}}_l T^F$ is independent of θ, and is given by an explicit formula involving the Green functions of reductive F-stable subgroups of G^F associated with the maximal torus T.

We first recall the character formula for the virtual characters $R^G_{T,\theta}$ ([5, Theorem 4.2; 12, Prop. 2.13.1]). Let s, $u \in G^F$, with s semisimple, u unipotent, and $su = us$. If $g = su$, we use the notation $g_{ss} = s$ and $g_{uni} = u$. We have

$$R^G_{T,\theta}(us) = |C_G(s)^{\circ F}|^{-1} \sum_{\substack{g \in G^F \\ gsg^{-1} \in T^F}} Q_T^{C_G(gsg^{-1})^\circ}(gug^{-1})\,\theta(gsg^{-1}), \tag{4.1}$$

where $Q_T^{C_G(gsg^{-1})^\circ}$ is the Green function of the connected reductive group $C_G(gsg^{-1})^\circ$ associated with the maximal torus T.

(4.2) THEOREM. *Let Γ_z, $\mathcal{H}_z$, and (T,θ) be as in Theorem* 3.1. *There exists a unique homomorphism of algebras $f_{T,z}\colon \mathcal{H}_z \to \bar{\mathbb{Q}}_l T^F$, independent of θ, with the property that each homomorphism $f_{T,\theta,z}\colon \mathcal{H}_z \to \bar{\mathbb{Q}}_l$, for $\theta \in \operatorname{Irr} T^F$, can be factored as*

$$f_{T,\theta,z} = \hat{\theta} \circ f_{T,z},$$

where $\hat{\theta}$ is the extension of θ to a representation of $\bar{\mathbb{Q}}_l T^F$. Let $f_{T,z}(\mathbf{c}) = \sum f_{T,z}(\mathbf{c})(t)\, t \in \bar{\mathbb{Q}}_l T^F$, for $\mathbf{c} \in \mathcal{H}_z$. The value of the coefficient function $f_{T,z}(\mathbf{c}_{n,z})(t)$, for a standard basis element $\mathbf{c}_{n,z}$ of $\mathcal{H}_z$ and $t \in T^F$, is given by the following formula, involving the Green function $Q_T^{C_G(t)^\circ}$ and the linear character ψ_z of U_0^F such that $\Gamma_z = \psi_z^{G^F}$:

$$\begin{aligned} f_{T,z}(\mathbf{c}_{n,z})(t) = {} & \operatorname{ind} n \langle Q_T^G, \Gamma_z\rangle^{-1}\, |U_0^F|^{-1}\, |C_G(t)^{\circ F}|^{-1} \\ & \times \sum_{\substack{g \in G^F,\, u \in U_0^F \\ (gung^{-1})_{ss} = t}} \psi_z(u^{-1})\, Q_T^{C_G(t)^\circ}((gung^{-1})_{uni}). \end{aligned} \tag{4.3}$$

Proof. We first prove that, for each standard basis element $\mathbf{c}_{n,z} = \operatorname{ind} n\mathbf{e}_{\psi_z} n\mathbf{e}_{\psi_z}$ of $\mathcal{H}_z$, we have

$$f_{T,\theta,z}(\mathbf{c}_{n,z}) = \sum_{t \in T^F} f_{T,z}(\mathbf{c}_{n,z})(t)\,\theta(t), \tag{4.4}$$

where $f_{T,z}\colon \mathcal{H}_z \to \bar{\mathbb{Q}}_l T^F$ is a $\bar{\mathbb{Q}}_l$-linear map, independent of θ, given by the formula (4.3). Since

$$\mathbf{a}_{T,\theta,z} = \mathbf{e}_z \cdot \mathbf{r}_{T,\theta} = \sum \alpha_{T,\theta,z}(g^{-1})\, g,$$

we have

$$\alpha_{T,\theta,z}(1) = \langle \psi_z, R^G_{T,\theta} \mid_{U_0^F} \rangle = \langle \Gamma_z, R^G_{T,\theta} \rangle \neq 0$$

by Theorem 2.1 and Frobenius reciprocity. Since Γ_z is supported on the set of unipotent elements in G^F, and $R^G_{T,\theta}(u) = Q^G_T(u)$ for each unipotent element u, we have

$$\langle \Gamma_z, R^G_{T,\theta} \rangle = \langle \Gamma_z, Q^G_T \rangle,$$

which is independent of θ.

We next observe that, by (3.1ii) and the previous remark,

$$f_{T,\theta,z}(\mathbf{c}_{n,z}) = \langle Q^G_T, \Gamma_z \rangle^{-1} \gamma_{n,\theta,z},$$

where $\gamma_{n,\theta,z}$ is the coefficient of the identity element 1 in $\mathbf{c}_{n,z}\mathbf{a}_{T,\theta,z}$. Since $\mathbf{e}_z$ is the identity element of $\mathscr{H}_z$,

$$\mathbf{c}_{n,z}\mathbf{a}_{T,\theta,z} = \mathbf{c}_{n,z}\mathbf{e}_z\mathbf{r}_{T,\theta} = \mathbf{c}_{n,z}\mathbf{r}_{T,\theta}.$$

Moreover, by [4, (11.31)],

$$\mathbf{c}_{n,z} = |U_0^F|^{-1} \sum_{u_1 n u_2 \in U_0^F n U_0^F} \psi_z(u_1^{-1} u_2^{-1})\, u_1 n u_2,$$

and $\mathbf{r}_{T,\theta} = \sum_g R^G_{T,\theta}(g^{-1})\, g$, by definition. It follows that

$$\begin{aligned}\gamma_{n,\theta,z} &= |U_0^F|^{-1} \sum_{u_1 n u_2 \in U_0^F n U_0^F} \psi_z(u_1^{-1} u_2^{-1})\, R^G_{T,\theta}(u_1 n u_2) \\ &= \operatorname{ind} n\, |U_0^F|^{-1} \sum_{u \in U_0^F} \psi_z(u^{-1})\, R^G_{T,\theta}(un),\end{aligned}$$

since $R^G_{T,\theta}$ is a class function, and the double coset $U_0^F n U_0^F$ contains ind n one-sided cosets of U_0^F. We now obtain, by the character formula (4.1) and the preceding calculation,

$$\begin{aligned}f_{T,\theta,z}(\mathbf{c}_{n,z}) = \operatorname{ind} n \langle Q^G_T, \Gamma_z \rangle^{-1} |U_0^F|^{-1} &\sum_{t \in T^F} |C_G(t)^{\circ F}|^{-1} \\ \times &\sum_{\substack{g \in G^F u \in U_0^F \\ (gung^{-1})_{ss} = t}} \psi_z(u^{-1})\, Q_T^{C_G(t)^\circ}((gung^{-1})_{uni})\, \theta(t).\end{aligned}$$

This proves (4.4), with $f_{T,z}(\mathbf{c}_{n,z})(t)$ as in (4.3).

If $f'_{T,z}\colon \mathscr{H}_z \to \bar{\mathbb{Q}}_l T^F$ is another linear map satisfying (4.4), then for $\mathbf{c} \in \mathscr{H}_z$,

$$\sum_{t \in T^F} f_{T,z}(\mathbf{c})(t)\, \theta(t) = \sum_{t \in T^F} f'_{T,z}(\mathbf{c})(t)\, \theta(t)$$

for all irreducible characters θ of T^F. This implies that $f_{T,z}(\mathbf{c}) = f'_{T,z}(\mathbf{c})$ for all $\mathbf{c} \in \mathscr{H}_z$, so the coefficient functions $f_{T,z}(\mathbf{c})(t)$ given in (4.4) are uniquely determined.

It remains to prove that $f_{T,z}$ is a homomorphism of algebras. We have $f_{T,\theta,z}(\mathbf{cc}') = f_{T,\theta,z}(\mathbf{c})\, f_{T,\theta,z}(\mathbf{c}')$, for all characters θ of T^F, and elements $\mathbf{c}$, $\mathbf{c}' \in \mathscr{H}_z$. By the factorization (4.4), we obtain

$$\hat{\theta}(f_{T,z}(\mathbf{cc}')) = \hat{\theta}(f_{T,z}(\mathbf{c})\, f_{T,z}(\mathbf{c}')),$$

for all characters θ. It follows that $f_{T,z}(\mathbf{cc}') = f_{T,z}(\mathbf{c})\, f_{T,z}(\mathbf{c}')$, completing the proof.

Remarks. Homomorphisms from $\mathscr{H}_z$ to the group algebra $\bar{\mathbb{Q}}_l T^F$ are by no means unique. For example, let $\{h_{\theta,z}\}$ be an arbitrary set of homomorphisms from $\mathscr{H}_z$ to $\bar{\mathbb{Q}}_l$, indexed by the set of irreducible characters θ of T^F (such as $\{f_{T,\theta,z}\}$ or some permutation of this set). Let $\{(\bar{\mathbb{Q}}_l)_\theta\}$ be a set of copies $\bar{\mathbb{Q}}_l$ indexed by the characters θ. Then $\Pi_\theta h_{\theta,z}\colon \mathscr{H}_z \to \Pi(\bar{\mathbb{Q}}_l)_\theta$ is a homomorphism of algebras. Moreover

$$\Pi\hat{\theta}\colon \bar{\mathbb{Q}}_l T^F \to \Pi(\bar{\mathbb{Q}}_l)_\theta$$

is an isomorphism of algbras. It follows that there exists a unique homomorphism of algebras $h_{T,z}\colon \mathscr{H}_z \to \bar{Q}_l T^F$ such that the following diagram commutes:

$$\begin{tikzcd}
\bar{\mathbb{Q}}_l T^F \arrow[rr, "\Pi\hat{\theta}"] & & \Pi(\bar{\mathbb{Q}}_l)_\theta \\
& \mathscr{H}_z \arrow[ul, "h_{T,z}"] \arrow[ur, "\Pi h_{\theta,z}"'] &
\end{tikzcd}$$

Moreover, $h_{T,z}$ has the factorization property

$$h_{\theta,z} = \hat{\theta} \circ h_{T,z}$$

for each irreducible character θ. (I am indebted to J. C. Jantzen for this observation).

The formula (4.3) singles out the unique homomorphism $f_{T,z}$ such that for each $\theta \in \operatorname{Irr} T^F$, the representation $\hat{\theta} \circ f_{T,z}$ of $\mathscr{H}_z$ corresponds to the unique common irreducible constituent of Γ_z and $R^G_{T,\theta}$.

Another test of whether a homomorphism $h_{T,z}\colon \mathscr{H}_z \to \bar{\mathbb{Q}}_l T^F$ coincides with $f_{T,z}$, which does not involve the Green functions $\{Q^G_T\}$ directly, is given as follows.

(4.5) Proposition. *Let* $h_{T,z}\colon \mathscr{H}_z \to \bar{\mathbb{Q}}_l T^F$ *be a homomorphism of algebras. For each* $\theta \in T^F$, $\hat{\theta} \circ h_{T,z}$ *is an irreducible representation of* $\mathscr{H}_z$ *corresponding to an irreducible character* χ_θ *of* G^F, *by* (2.2i), *whose values are given by* (2.2v). *Then* $h_{T,z} = f_{T,z}$ *if and only if* $\chi_\theta = \chi_{T,\theta,z}$, *where* $\chi_{T,\theta,z}$ *is the unique common constituent of* $R^G_{T,\theta}$ *and* Γ_z, *given by Theorem* 2.1.

The proof is immediate from Theorems 2.1, 3.1 and Proposition 2.2.

5. Example: $G = SL_2$

Let $G = SL_2$, with its usual BN-pair, $\mathbb{F}_q$-structure, and Frobenius map F. Then $G^F = SL_2(\mathbb{F}_q)$. There are two G^F-conjugacy classes of maximal tori T, parametrized by the elements $\{1, s\}$ of the Weyl group of G. As representatives of T^F, we take

$$T_1^F = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} : \mu \in \mathbb{F}_q^{\cdot} \right\}, \qquad \text{with} \quad |T_1^F| = q-1,$$

and the Coxeter torus

$$T_s^F \cong C,$$

where

$$C = \{\xi \in \mathbb{F}_{q^2}^{\cdot} : \xi^{q+1} = 1\}, \qquad \text{and} \qquad |T_s^F| = |C| = q+1.$$

If q is odd, we have $|Z(G)/Z(G)^\circ| = 2$, and there are two Gelfand–Graev characters, while if q is even, $Z(G) = 1$, and there is just one Gelfand–Graev character.

Each Gelfand–Graev character is given by

$$\Gamma = \psi^{G^F}, \qquad \psi \in \operatorname{Irr} U_0^F, \qquad \psi \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \chi(\alpha), \tag{5.1}$$

for a fixed nontrivial character χ of the additive group of $\mathbb{F}_q$. When q is odd, there are two T_0^F-conjugacy classes of nontrivial characters ψ of U_0^F, while if q is even, there is only one. For the rest of the section, Γ, ψ, and χ are fixed, as in (5.1). We let $\mathbf{e}_\psi$ be the idempotent in $\bar{\mathbb{Q}}_l U_0^F$ corresponding to ψ, and let

$$\mathscr{H} = \mathbf{e}_\psi \bar{\mathbb{Q}}_l G^F \mathbf{e}_\psi$$

be the Hecke algebra of $\Gamma = \psi^{G^F}$. The dimension of $\mathscr{H}$ is q or $q+1$, depending on whether q is even or odd. In either case, $\mathscr{H}$ has $q-1$ standard basis elements

$$\mathbf{c}_\lambda = q\mathbf{e}_\psi n_\lambda \mathbf{e}_\psi, \qquad n_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix},$$

together with the identity element $\mathbf{e}_\psi$, and one other basis element $\mathbf{c}_{-1} = \mathbf{e}_\psi(-1)\,\mathbf{e}_\psi$, where $(-1) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, in case q is odd.

(5.2) THEOREM. *The homomorphisms $f_T\colon \mathscr{H} \to \bar{\mathbb{Q}}_l T^F$, defined by* (4.3), *for the two classes of maximal F-stable tori T, are given as follows. Let*

$$f_T(\mathbf{c}) = \sum_{t \in T^F} f_T(\mathbf{c})(t)\,t, \qquad \text{for} \quad \mathbf{c} \in \mathscr{H}.$$

Then, for each standard basis element $\mathbf{c}_\lambda$,

$$f_{T_1}(\mathbf{c}_\lambda)\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = \chi(\lambda(\mu+\mu^{-1})), \qquad \text{for} \quad \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \in T_1^F,$$

and

$$f_{T_s}(c_\lambda)(\xi) = -\chi(\lambda(\xi+\xi^{-1})), \qquad \text{for} \quad \xi \in C \cong T_s^F. \tag{5.3}$$

The homomorphism $f_{T_1,\theta} = \hat{\theta} \circ f_{T_1}$, for $\theta\left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right) = \pi(\mu)$, *with π a multiplicative character of $\mathbb{F}_q^\cdot$, is given by*

$$\hat{\theta} \circ f_{T_1}(\mathbf{c}_\lambda) = \sum_{\mu \in \mathbb{F}_q^\cdot} \chi(\lambda(\mu+\mu^{-1}))\,\pi(\mu). \tag{5.4}$$

These correspond to irreducible principal series characters of $SL_2(\mathbb{F}_q)$. An irreducible character θ' of *$T_s^F \cong C$ is given by a multiplicative character π' of C, and the corresponding representation of $\mathscr{H}$ by*

$$\hat{\theta}' \circ f_{T_s}(\mathbf{c}_\lambda) = -\sum_{\xi \in C} \chi(\lambda(\xi+\xi^{-1}))\,\pi'(\xi). \tag{5.5}$$

These correspond to cuspidal irreducible characters of $SL_2(\mathbb{F}_q)$ in the case $\theta' \neq 1$.

Proof. By (4.3), we have

$$f_T(\mathbf{c}_\lambda)(t) = \langle Q_T^G, \Gamma\rangle \operatorname{ind} n_\lambda\, |U_0^F|^{-1}\, |C_G(t)^{\circ F}|^{-1} \times \sum_{\substack{g \in G^F,\, u \in U_0^F \\ (gun_\lambda g^{-1})_{ss} = t}} \psi(u^{-1})\, Q_T^{C_G(t)^\circ}((gun_\lambda g^{-1})_{uni}).$$

for a maximal F-stable torus T.

First assume $T = T_1$, and let $t = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \in T_1^F$. Then, by Frobenius reciprocity,

$$\langle Q_T^G, \Gamma \rangle = \langle Q_T^G|_{U_0^F}, \psi \rangle = q^{-1} \sum_u Q_T^G(u)\,\psi(u^{-1}) = 1.$$

Here we have used the facts that $Q_T^G(1) = q+1$, and $Q_T^G(u) = 1$ for all $u \neq 1$ by [5, Sect. 9], since the nonidentity elements in U_0^F are regular. First assume that $t \neq \pm 1$. In that case, $C_G(t)^{0F} = T_1^F$, and $Q_{T_1}^{T_1} = 1$. Moreover, trace $x =$ trace x_{ss} for $x \in G^F$, and semisimple elements are determined up to conjugacy by their trace. It follows that, for $t \neq \pm 1$, the equation $(gun_\lambda g^{-1})_{ss} = t$ has $|C_G(t)^{0F}|$ solutions, with each solution satisfying trace $un_\lambda =$ trace $t = \mu + \mu^{-1}$. Upon substituting this information in the formula above, we obtain $f_{T_1}(\mathbf{c}_\lambda)(t) = \chi(\lambda(\mu + \mu^{-1}))$. If $t = \pm 1$, then $C_G(t)^0 = G$, and a similar argument applies.

Now let $T^F = T_s^F \cong C$. By [5, Theorem 7.1], we have

$$\langle Q_T^G, \Gamma \rangle = q^{-1}\left(Q_T(1) + \sum_{u \neq 1} \psi(u^{-1}) \right) = q^{-1}(-q+1-1) = -1.$$

For $t \in T_s^F$, trace $t = \xi + \xi^{-1}$ if t corresponds to $\xi \in C$. The other steps are the same as in the preceding case, and we obtain $f_{T_s}(\mathbf{c}_\lambda)(t) = -\chi(\lambda(\xi + \xi^{-1}))$ if $t \to \xi \in C$, for $t \in T_s^F$.

Let $\theta \in \operatorname{Irr} T_1^F$; then $\theta\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = \pi(\mu)$ for a multiplicative character π of $\mathbb{F}_q^\cdot$, and we obtain the formula for $f_{T_1,\theta} = \hat{\theta} \circ f_{T_1}$ given in the statement of the theorem. It is easily checked that these homomorphisms are the restrictions to $\mathscr{H}$ of the irreducible principal series characters $\chi_{T_1,\theta}$.

The characters θ' of $T_s^F \cong C$ correspond to multiplicative characters π' of C, and the representations $f_{T_s,\theta'} = \hat{\theta}' \circ f_{T_s}$ are given by the formula (5.4). It remains to verify that the corresponding irreducible characters of G^F are cuspidal. For this, it is sufficient to prove that $\langle \chi_{T_s,\theta'}, (1_{U_0^F})^{G^F} \rangle = 0$. As $(1_{U_0^F})^{G^F}$ is the sum of the irreducible principal series characters, the result will follow when it is shown that the irreducible characters $f_{T_1,\theta}$ and $f_{T_s,\theta'}$ of $\mathscr{H}$ are orthogonal with respect to the standard dual basis of $\mathscr{H}$ [4, (11.30iii)]. This is readily shown. Alternatively, the result follows from [5, Theorem 8.3] if θ' is in general position, and by restriction from GL_2 to SL_2, in case θ' has order 2.

Remark. The formulas (5.4) and (5.5) were first stated (without proof) by Gelfand and Graev in [9]. They noted that the functions $\varphi(\lambda)$ on F_q defined by the formulas (5.4) and (5.5) are analogous to the integral formulas for Bessel functions over $\mathbb{C}$, and called them, in this case, *Bessel functions over finite fields.* The homomorphisms $f_{T_s,\theta'}$ can be derived from the construction of the discrete series representations of $SL_2(\mathbb{F}_q)$ by Gelfand *et al.* [10].

We conclude this section with a direct proof that the map $f_{T_s}: \mathscr{H} \to \bar{Q}_l T_s^F$ defined by (5.3) is indeed a homomorphism of algebras. The argument is independent of Deligne–Lusztig theory, and is based instead on identities for exponential sums. It goes almost without saying that analogous identities for higher rank groups, and their application to the direct construction of the homomorphisms f_T, are a main goal of this approach to representations of $\mathscr{H}$ in the first place. In this connection see Chang's paper [2] on the representations of the Hecke algebra $\mathscr{H}$ of the Gelfand–Graev representation, in case $G = GL_3$.

The identities we require are the following ones.

(5.6) LEMMA (Chang [2]). *Let χ be a nontrivial additive character of $\mathbb{F}_q$. Then we have*

(i) $\sum_{\lambda \in \mathbb{F}_q^\cdot} \chi(\alpha\lambda + \beta\lambda^{-1}) = \sum_{\lambda \in \mathbb{F}_q^\cdot} \chi(\lambda + \alpha\beta\lambda^{-1})$, *for all* $\alpha, \beta \in \mathbb{F}_q^\cdot$.

(ii) $\sum_{\xi \in C} \chi(\gamma\xi + \gamma^q\xi^q) = -\sum_{\lambda \in \mathbb{F}_q^\cdot} \chi(\lambda + \gamma\gamma^q\lambda^{-1})$, *for all* $\gamma \in \mathbb{F}_{q^2}^\cdot$, *and* $C = \{\xi \in \mathbb{F}_{q^2}^\cdot : \xi^{q+1}\} = 1$.

The identity (i) is proved by a simple change of variables. Chang's proof of (ii) in [2] is based on an analysis of quadratic equations over $\mathbb{F}_q$. It also follows from the Davenport–Hasse identity for Gauss sums [14].

The main point in checking that the maps f_{T_s} given in (5.3) are homomorphisms is as follows.

(5.7) PROPOSITION. *Let $[\mathbf{c}_\lambda \mathbf{c}_{\lambda'} : \mathbf{c}_{\lambda''}]$ be the structure constant of $\mathscr{H}$ involving the standard basis elements $\mathbf{c}_\lambda$, $\mathbf{c}_{\lambda'}$, $\mathbf{c}_{\lambda''}$, for $\lambda, \lambda', \lambda'' \in \mathbb{F}_q^\cdot$. Let $f(\mathbf{c}_\lambda)$ be the element of $\bar{\mathbb{Q}}_l C$ whose coefficient at $\xi \in C$ is given by*

$$f(\mathbf{c}_\lambda)(\xi) = -\chi(\lambda(\xi + \xi^{-1})).$$

Then

$$\sum_{\eta \in C} f(\mathbf{c}_\lambda)(\xi\eta)\, f(\mathbf{c}_{\lambda'})(\eta^{-1})$$

$$= \sum_{\lambda'' \in \mathbb{F}_q^\cdot} [\mathbf{c}_\lambda \mathbf{c}_{\lambda'} : \mathbf{c}_{\lambda''}]\, f(\mathbf{c}_{\lambda''})(\xi), \qquad \textit{for all} \quad \xi \in C,\ \xi \neq \pm 1.$$

Proof. By computing the structure constant $[\mathbf{c}_\lambda \mathbf{c}_{\lambda'} : \mathbf{c}_{\lambda''}]$ as in [4, (11.30ii)], we obtain

$$[\mathbf{c}_\lambda \mathbf{c}_{\lambda'} : \mathbf{c}_{\lambda''}] = \chi(\lambda\lambda'\lambda''^{-1} + \lambda\lambda''\lambda'^{-1} + \lambda'\lambda''\lambda^{-1}).$$

By the definition of $f(\mathbf{c}_\lambda)$ given in the statement of the Proposition, we have, for all $\xi \in C$, $\xi \notin \mathbb{F}_q$,

$$\begin{aligned}\sum_{\eta \in C} f(\mathbf{c}_\lambda)(\xi\eta)\, f(\mathbf{c}_{\lambda'})(\eta^{-1}) &= \sum_{\eta \in C} \chi(\lambda(\xi\eta + \xi^{-1}\eta^{-1}))\, \chi(\lambda'(\eta + \eta^{-1}))\\ &= \sum_{\eta \in C} \chi((\lambda\xi + \lambda')\eta + (\lambda\xi^q + \lambda')\,\eta^q)\\ &= -\sum_{\mu \in \mathbb{F}_q^{\cdot}} \chi(\mu + (\lambda\xi + \lambda')(\lambda\xi^q + \lambda')\,\mu^{-1}),\end{aligned}$$

by Lemma 5.6ii and the fact that $\eta^{-1} = \eta^q$ since $\eta \in C$. The last expression is equal to

$$-\sum_{\lambda'' \in \mathbb{F}_q^{\cdot}} \chi(\lambda\lambda'\lambda''^{-1} + \lambda\lambda''\lambda'^{-1} + \lambda'\lambda''\lambda^{-1})\, \chi(\lambda''(\xi + \xi^{-1}))$$

by the first part of the lemma. If $\xi \in \mathbb{F}_q$, then $\xi = \pm 1$, and the possibility that $\lambda\xi + \lambda' = 0$ occurs. In this case a more general version of Lemma 5.6 is required (see [2]). We omit the discussion of this situation.

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